

TMDs on the Lattice

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in collaboration with

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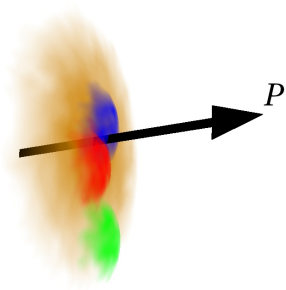
Andreas Schäfer (Univ. Regensburg),

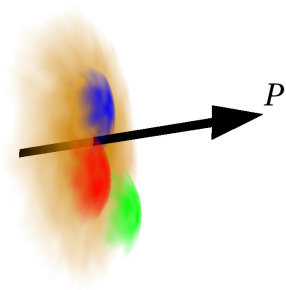
and the LHP Collaboration

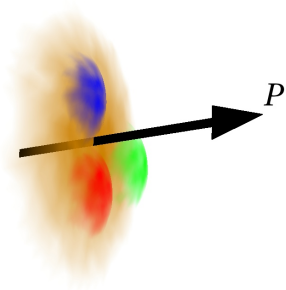
[HÄGLER ET AL. EPL88 61001 (2009)]

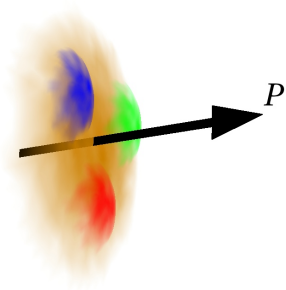
[MUSCH arXiv:0907.2381]

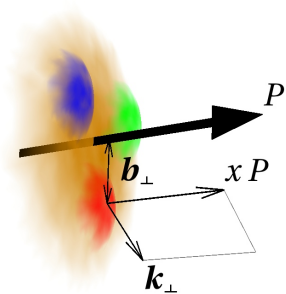


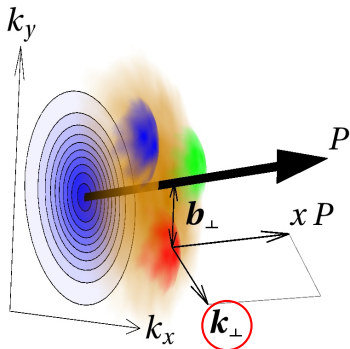












TMDs

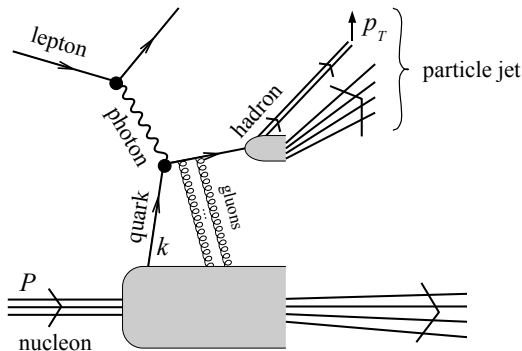
transverse **m**omentum dependent
parton distribution functions

e.g., $f_1(x, \mathbf{k}_\perp^2)$

\Rightarrow quark density $\rho(\mathbf{k}_\perp)$.

- x (longitudinal momentum fraction) \Rightarrow PDFs
- x, \mathbf{b}_\perp (impact parameter) \Rightarrow GPDs
- x, \mathbf{k}_\perp (intrinsic transverse momentum) \Rightarrow TMDs

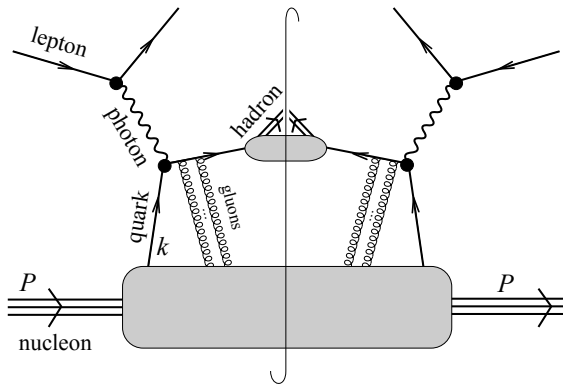
e.g., semi-inclusive DIS [COLLINS PLB 93], [BACCHETTA ET AL. JHEP 07]



experiments sensitive to TMDs

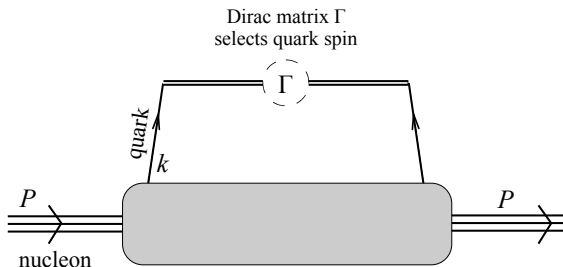
COMPASS (CERN), HERMES (DESY), JLab, RHIC (BNL), Fermilab, also planned at J-PARC, FAIR (GSI), NICA (JINR), ..., EIC (BNL/JLab?)

e.g., semi-inclusive DIS [COLLINS PLB 93], [BACCHETTA ET AL. JHEP 07]



$$\frac{d\sigma}{d^3 P_h d^3 P_v} \propto \underbrace{H(Q^2, \dots)}_{\text{hard part}} \int d^2 \mathbf{k}_\perp \underbrace{f_1(x, \mathbf{k}_\perp, \dots)}_{\text{TMD}} \underbrace{D_h(z, \mathbf{k}_\perp + \mathbf{q}_\perp, \dots)}_{\text{fragmentation f.}}$$

(no soft factor taken into account, see [JI, MA, YUAN PRD 71 (2005)])



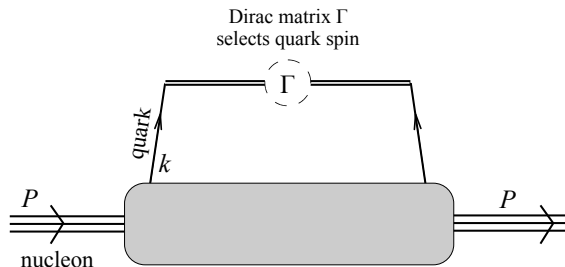
$$\Phi^{[\Gamma]}(k, P, S) \equiv \langle P, S | \bar{q}(k) \Gamma q(k) | P, S \rangle$$

lightcone coord. $w^\pm = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$, so $w = w^+ \hat{n}_+ + w^- \hat{n}_- + w_\perp$
 proton flies along z-axis: P^+ large, $P_\perp = 0$

parametrization in terms of TMDs, example

$$\int dk^- \Phi^{[\gamma^+]}(k, P, S) \Big|_{k^+ = xP^+} = f_1(x, \mathbf{k}_\perp^2) - \frac{\epsilon_{ij} k_i \mathbf{S}_j}{m_N} f_{1T}(x, \mathbf{k}_\perp)$$

[RALSTON, SOPER NPB 1979], [MULDERS, TANGERMAN NPB 1996], [GOEKE, METZ, SCHLEGEL PLB 2005]



$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4 \ell}{(2\pi)^4} e^{-ik \cdot \ell} \langle P, S | \bar{q}(\ell) \Gamma \mathcal{U}_q(0) | P, S \rangle$$

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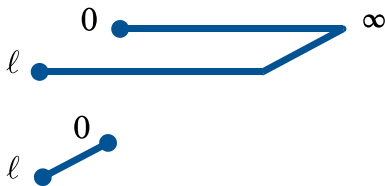
$\langle P | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P \rangle$ is gauge invariant.

continuum

$$\mathcal{U} \equiv \mathcal{P} \exp \left(-ig \int_0^\ell d\xi^\mu A_\mu(\xi) \right)$$

along path from 0 to ℓ

- factorization in SIDIS :
path runs to infinity and back
- simplification*:
straight path (for first studies)



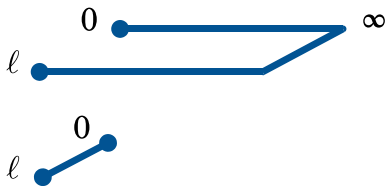
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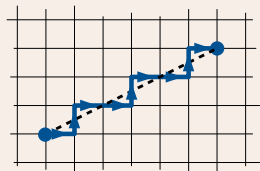
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along path from 0 to ℓ

lattice

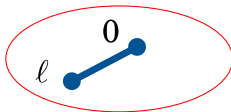


product of link variables

- factorization in SIDIS :
path runs to infinity and back



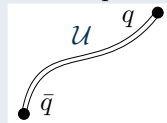
- simplification:**
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continuum

[CRAIGIE, DORN NPB185,204 (1981)]

smooth path



$$[\bar{q} \mathcal{U} q]_{\text{ren}} = Z^{-1} \exp\left(-\delta\hat{m} \frac{l}{a}\right) [\bar{q} \mathcal{U} q]$$

$\delta\hat{m}$: removes the length dependent renorm. factor

static quark potential

$$V_{\text{ren}}(r) = V(r) + 2\delta\hat{m}/a$$

string [LÜSCHER, SYMANZIK, WEISZ (1980)]

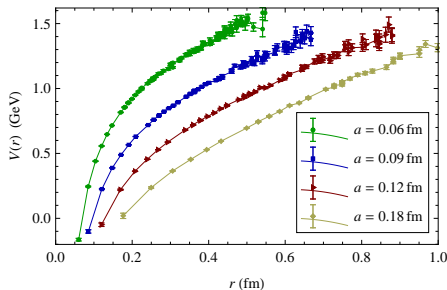
at large r : $V_{\text{ren}}(r) \approx$

$$V_{\text{string}}(r) = \sigma r - \pi/12r + 0$$

method [CHENG PRD77,014511 (2008)]

determine $\delta\hat{m}$ from

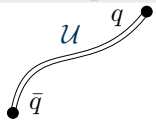
$$V_{\text{ren}}(0.7 \text{ fm}) \stackrel{!}{=} V_{\text{string}}(0.7 \text{ fm})$$



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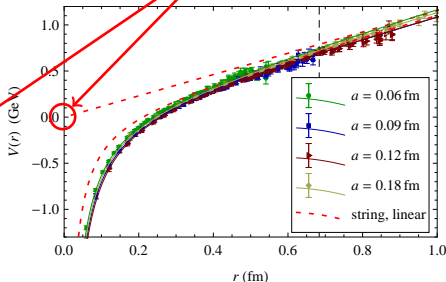
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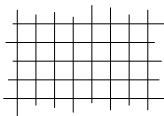
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renormalization condition $C^{\text{ren}} = 0$ 

We employ the Chroma library [EDWARDS, JOO (2005)] to process



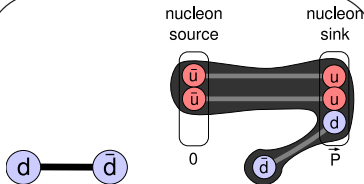
MILC gauge configurations

staggered Asqtad action,
 2+1 flavors, $a \approx 0.124$ fm,
 $m_\pi \approx 500, 610,$ and 760 MeV

[ORGINOS, TOUSSAINT PRD (1999)]

+ finer MILC lattices
 to test renormalization

[AUBIN ET AL. PRD (2004)]
 [BAZAVOV ET AL. 0903.3598]



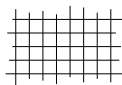
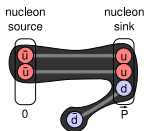
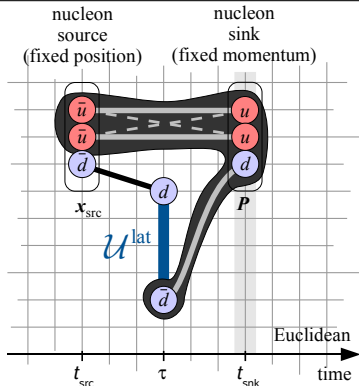
LHPC propagators

domain wall valence fermions,
 m_π adjusted to staggered sea,
 nucleon momenta:

$$\mathbf{P} = 0 \text{ and } |\mathbf{P}| = 500 \text{ MeV}$$

e.g., [HÄGLER ET AL. PRD (2008)]

Ingredients

Output : 3-point correlator $C_{3\text{pt}}$ gauge
configs.quark
propagatorsnucleon
sequential
propagatorsform ratio $C_{3\text{pt}}/C_{2\text{pt}}$, take plateau

$$\Rightarrow \langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle$$

[We neglect “disconnected contributions” (absent for up minus down).]

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isolation of Lorentz-invariant amplitudes

compare [MULDERS, TANGEMAN NPB (1996)]

$$\langle P, S | \bar{q}(\ell) \gamma_\mu \mathcal{U} q(0) | P, S \rangle = 4 \tilde{A}_2 P_\mu + 4i m_N^2 \tilde{A}_3 \ell_\mu$$

$$\begin{aligned} \langle P, S | \bar{q}(\ell) \gamma_\mu \gamma^5 \mathcal{U} q(0) | P, S \rangle &= -4 m_N \tilde{A}_6 S_\mu \\ &\quad -4i m_N \tilde{A}_7 P_\mu (\ell \cdot S) \\ &\quad +4 m_N^3 \tilde{A}_8 \ell_\mu (\ell \cdot S) \end{aligned}$$

$$\langle P, S | \bar{q}(\ell) \dots \mathcal{U} q(0) | P, S \rangle = \text{further structures (9 amplitudes in total)}$$

Transformation properties of the matrix element (\dagger , \mathcal{P} , \mathcal{T}) limit number of allowed structures. No \mathcal{T} -odd structures (Sivers function, ...) with straight gauge link.

The amplitudes fulfill $\tilde{A}_i(\ell^2, \ell \cdot P) = \left[\tilde{A}_i(\ell^2, -\ell \cdot P) \right]^*$.

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$\Rightarrow f_1(x, \mathbf{k}_\perp^2)$

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$$-4i m_N \tilde{A}_7 P_\mu (\ell \cdot S)$$

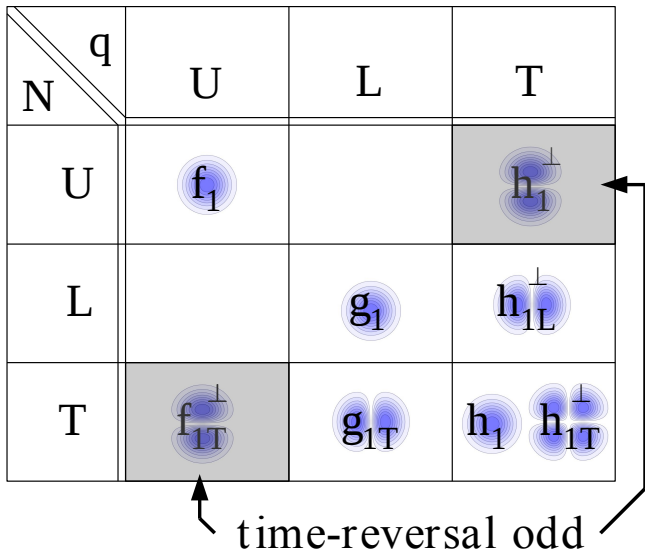
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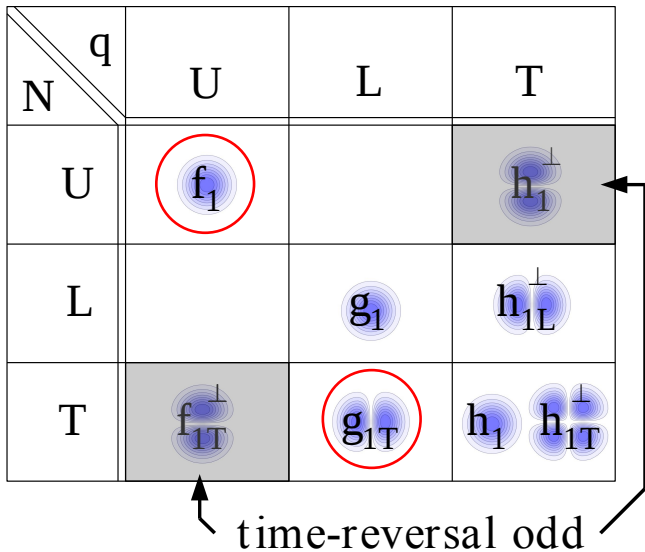
$$\Rightarrow g_{1T}(x, \mathbf{k}_\perp^2)$$

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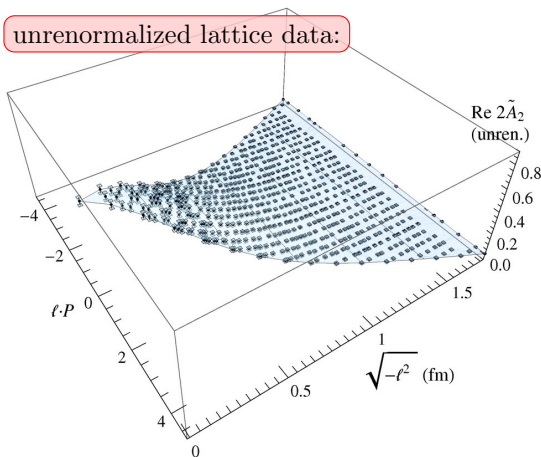


extract Lorentz-invariant amplitudes $\tilde{A}_i(\ell^2, \ell \cdot P)$, example :

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$$f_1(x, \mathbf{k}_\perp^2) = \int \frac{d(\ell \cdot P)}{2\pi} e^{ix(\ell \cdot P)} \int \frac{d^2 \ell_\perp}{(2\pi)^2} e^{-i\mathbf{k}_\perp \cdot \ell_\perp} 2\tilde{A}_2(\ell^2, \ell \cdot P) \Big|_{\ell^+=0}$$

unrenormalized lattice data:



$$\ell^2 \xleftrightarrow{\text{FT}} \mathbf{k}_\perp^2$$

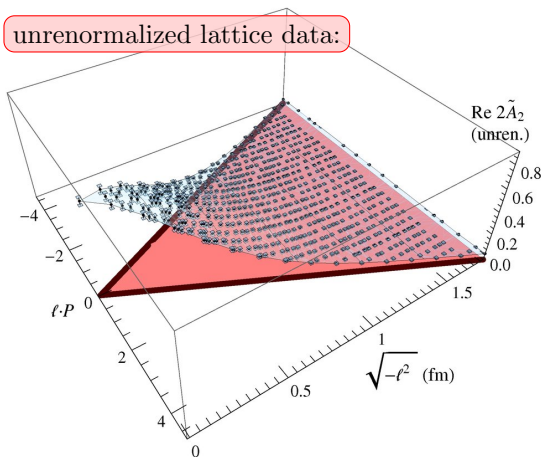
$$\ell \cdot P \xleftrightarrow{\text{FT}} x$$

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Euclidean lattice

$$\ell^0 = \ell_4 = 0$$

$$\Downarrow$$

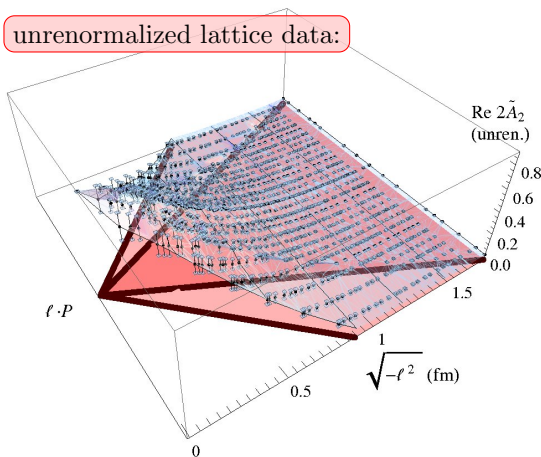
$$\ell^2 \leq 0, \\ |\ell \cdot P| \leq |\mathbf{P}| \sqrt{-\ell^2}$$

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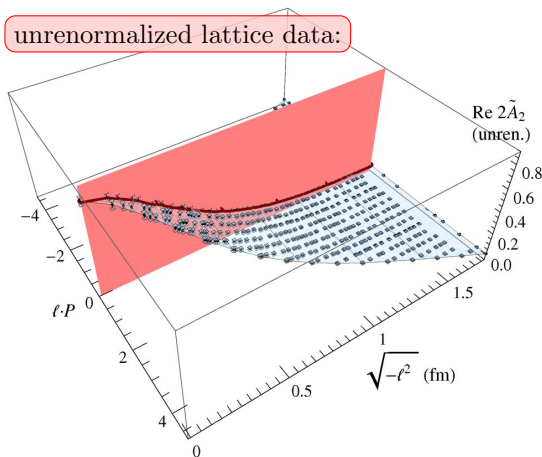
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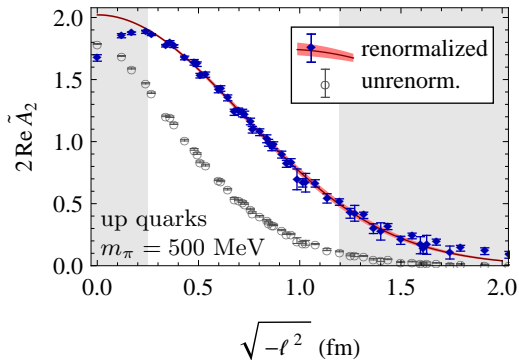
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$$\Downarrow$$

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$$f_1^{[1]}(\mathbf{k}_\perp^2) \equiv \int_{-1}^1 dx f_1(x, \mathbf{k}_\perp^2) = \int \frac{d^2 \ell_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot \ell_\perp} 2 \tilde{A}_2(-\ell_\perp^2, 0)$$



fit function

$$C_1 \exp(-|\ell|^2/\sigma_1^2)$$

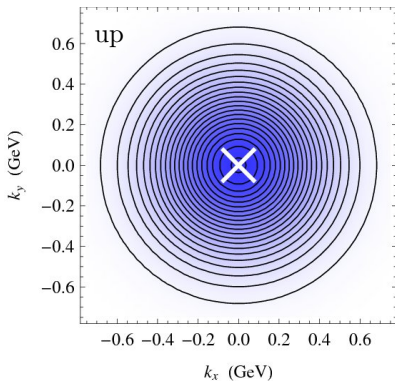
Z-factor

$$Z^{-1} C_1^{\text{up-down}} \stackrel{!}{=} 1$$

multiplicative
renormalization based on
quark counting

Density of unpolarized quarks (minus antiquarks)
in an unpolarized nucleon as a function of transverse momentum \mathbf{k}_\perp :

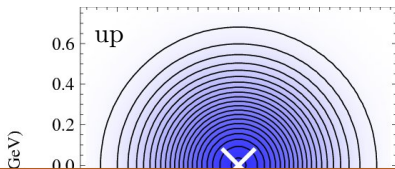
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axially symmetric

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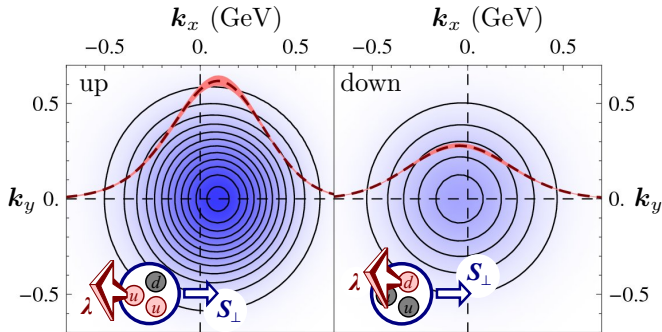
keep in mind



- correlator with straight Wilson line (“sW”)
- renormalized to string potential with $C = 0$
- Gaussian fit ansatz
(“wrong” at large- \mathbf{k}_\perp [DIEHL, arXiv:0811.0774])
- $m_\pi \approx 500$ MeV

Density of quarks with positive helicity, $\lambda = 1$,
in a transversely polarized nucleon, $\mathbf{S}_\perp = (1, 0)$:

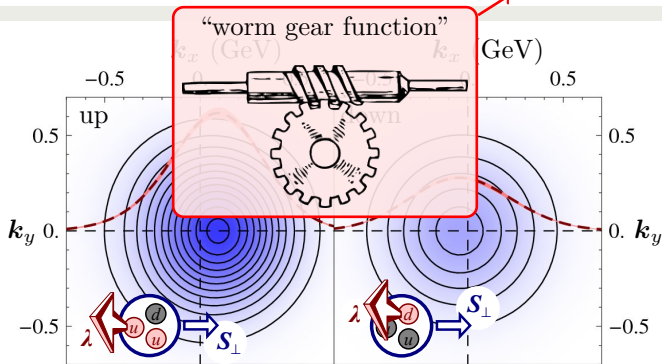
$$\begin{aligned} \rho_{TL}^{[1]}(\mathbf{k}_\perp; \mathbf{S}_\perp, \lambda) &\equiv \frac{1}{2} \int dx \int dk^- \Phi^{[\gamma^+ \frac{1}{2}(\mathbf{1} + \gamma^5)]}(k, P, S_\perp) \\ &= \frac{1}{2} f_1^{[1]}(\mathbf{k}_\perp^2) + \frac{\lambda}{2} \frac{\mathbf{k}_\perp \cdot \mathbf{S}_\perp}{m_N} g_{1T}^{[1]}(\mathbf{k}_\perp^2) \end{aligned}$$



($m_\pi \approx 500$ MeV, straight gauge link operator,
renormalization condition $C^{\text{ren}} = 0$, Gaussian fit)

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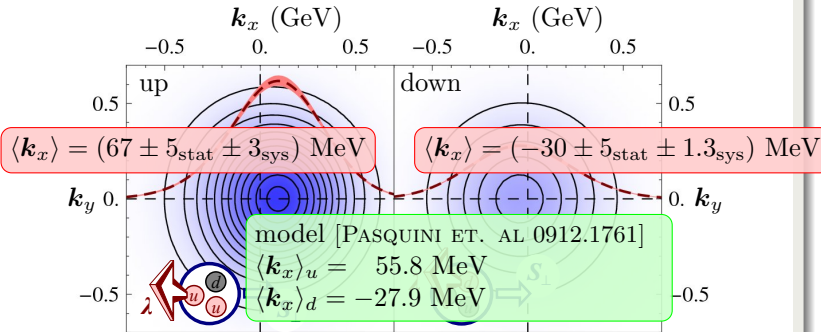
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Density of quarks with positive helicity, $\lambda = 1$,
in a transversely polarized nucleon, $\mathbf{S}_\perp = (1, 0)$:

$$\begin{aligned}\rho_{TL}^{[1]}(\mathbf{k}_\perp; \mathbf{S}_\perp, \lambda) &\equiv \frac{1}{2} \int dx \int dk^- \Phi^{[\gamma^+ \frac{1}{2}(\mathbf{1} + \gamma^5)]}(k, P, S_\perp) \\ &= \frac{1}{2} f_1^{[1]}(\mathbf{k}_\perp^2) + \frac{\lambda}{2} \frac{\mathbf{k}_\perp \cdot \mathbf{S}_\perp}{m_N} g_{1T}^{[1]}(\mathbf{k}_\perp^2)\end{aligned}$$



$(m_\pi \approx 500 \text{ MeV}, \text{ straight gauge link operator, renormalization condition } C^{\text{ren}} = 0, \text{ Gaussian fit})$

$$\frac{g_A}{g_V} \stackrel{\text{Gauss}}{\approx} \frac{\int dx \int d^2 \mathbf{k}_\perp g_1(x, \mathbf{k}_\perp)}{\int dx \int d^2 \mathbf{k}_\perp f_1(x, \mathbf{k}_\perp)} = \frac{-\tilde{A}_6^{\text{Gauss}}(0,0)}{\tilde{A}_2^{\text{Gauss}}(0,0)}$$

$$\frac{g_T}{g_V} \stackrel{\text{Gauss}}{\approx} \frac{\int dx \int d^2 \mathbf{k}_\perp h_1(x, \mathbf{k}_\perp)}{\int dx \int d^2 \mathbf{k}_\perp f_1(x, \mathbf{k}_\perp)} = \frac{-\tilde{A}_{9m}^{\text{Gauss}}(0,0)}{\tilde{A}_2^{\text{Gauss}}(0,0)}$$

lattice calculations at $m_\pi \approx 500$ MeV

| | our method (Gauss.) | standard method (local operators) |
|-------------|---------------------|--|
| $g_{A,u-d}$ | 1.19 ± 0.05 | 1.17 ± 0.03 [LHPC PRL96,052001 (2006)] |
| $g_{T,u-d}$ | 1.18 ± 0.04 | 1.06 ± 0.02 [LHPC PoS LAT2006, 121] ($\overline{\text{MS}}, \mu = 4 \text{ GeV}^2$) |

experiment:

$$g_{A,u-d} = 1.270 \pm 0.003 \quad [\text{PDG PLB667,1 (2008)}]$$

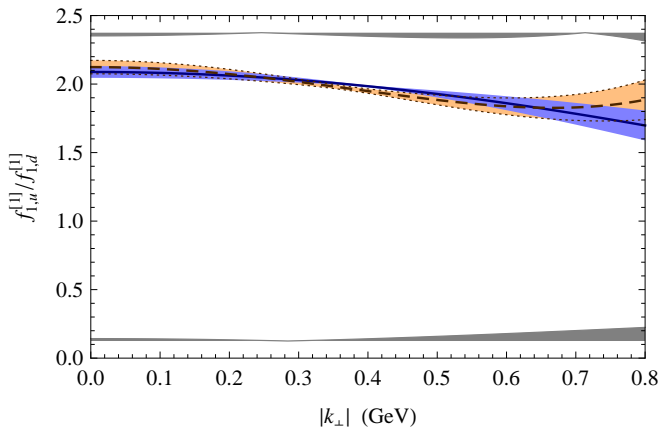
$$g_{T,u-d} = 0.77 \pm \sim 0.3 \text{ at } Q^2 = 0.8 \text{ GeV}^2$$

[ANSELMINO ET. AL. AIP Conf. Proc. (2009)]

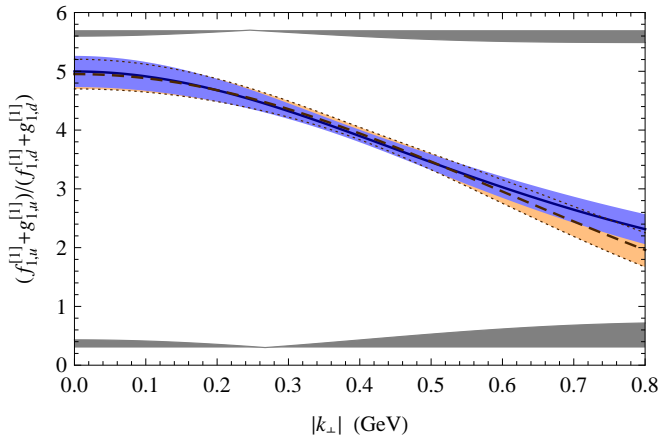


models ($\mu \approx 1 \text{ GeV}$), e.g.,

$$g_{T,u-d} = (0.6 - 1.2) \pm 0.2, \text{ depend. on } \langle \mathbf{k}_\perp^2 \rangle \quad [\text{GARY} \ \& \ \text{LEO PRL (2001)}]$$

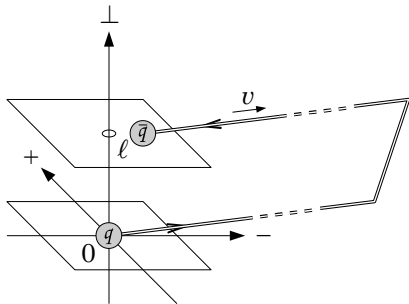


($m_{\pi} \approx 500$ MeV, straight gauge link operator,
 renormalization condition $C^{\text{ren}} = 0$, Gaussian fit)



$\left(\begin{array}{l} m_{\pi} \approx 500 \text{ MeV, straight gauge link operator,} \\ \text{renormalization condition } C^{\text{ren}} = 0, \text{ Gaussian fit} \end{array} \right)$

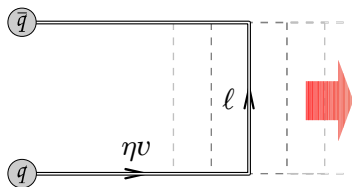
- ... appear in factorized SIDIS / Drell-Yan process
- are responsible for “time-reversal-odd” TMDs, such as f_{1T}^\perp (Sivers-function)



- gauge link = effective representation of struck quark (“final state interaction”)
- \Rightarrow (almost lightlike)

$$\zeta \equiv \frac{(v \cdot P)^2}{v^2} \rightarrow \pm \infty$$

- keep ζ finite to avoid “rapidity divergences”
- evolution equation in ζ [COLLINS, SOPER NPB (1981)]

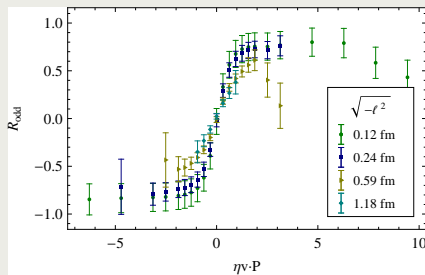


- v spatial $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |P_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes

[GOEKE, METZ, SCHLEGEL PLB (2005)]

$$\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta), \tilde{b}_i(\dots)$$

Test calculation: a time reversal odd ratio of amplitudes



$$R_{\text{odd}} = -\frac{\tilde{a}_{12} - \left(\eta \frac{m_N^2 v_1}{P_1}\right) \tilde{b}_8}{\tilde{a}_2}$$

Plateaus visible at large $|\eta|$.

“Time-reversal odd” \leftrightarrow
odd in $\eta v \cdot P$.

Part of the effect comes from the Sivers function f_{1T}^\perp !

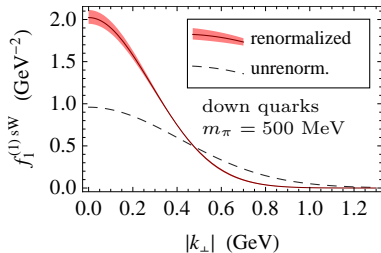
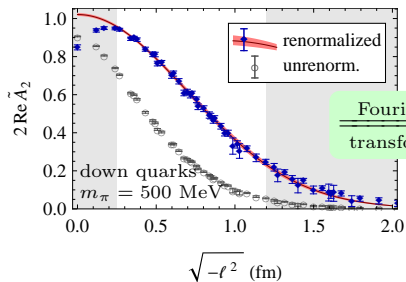
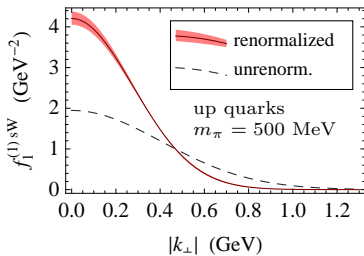
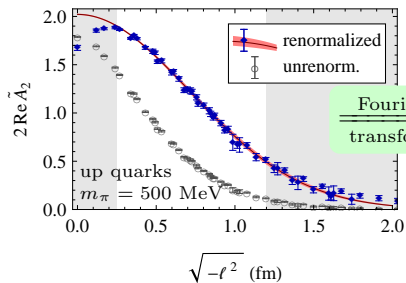
Summary:

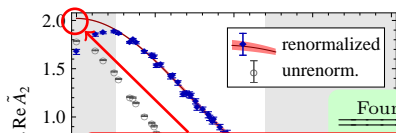
- We have explored ways to calculate intrinsic transverse momentum distributions in the nucleon with lattice QCD. We directly implement non-local operators on the lattice.
- First results are based on a simplified operator geometry (direct gauge link) and a Gaussian fit model, at $m_\pi \approx 500$ MeV:
 - We calculate the first Mellin moment of leading twist TMDs $f_1^{[1]}(\mathbf{k}_\perp^2)$, $g_{1T}^{[1]}(\mathbf{k}_\perp^2)$, $h_{1L}^{\perp[1]}(\mathbf{k}_\perp^2)$ etc.
 - \mathbf{k}_\perp -densities of longitudinally polarized quarks in a transversely polarized proton are deformed, due to non-vanishing $g_{1T}^{[1]}$.

Outlook:

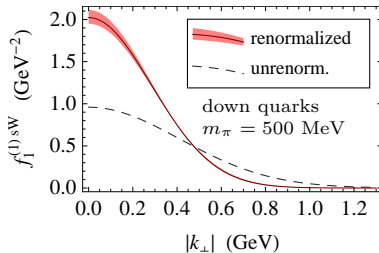
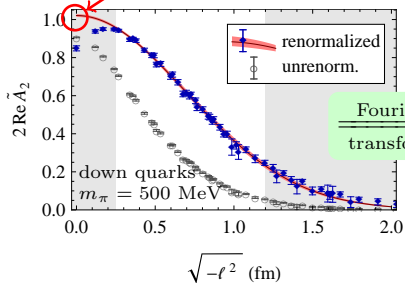
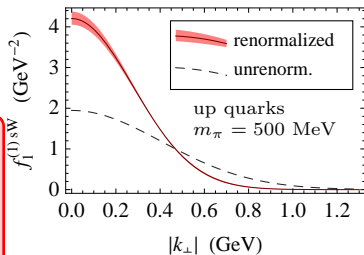
- Study of non-straight gauge links similar as in SIDIS.
- Beyond Gaussian fits:
Matching to perturbative behavior at small ℓ , i.e., large \mathbf{k}_\perp .

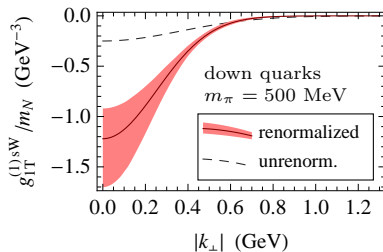
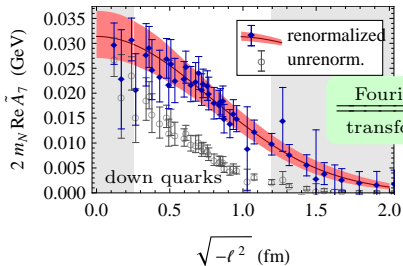
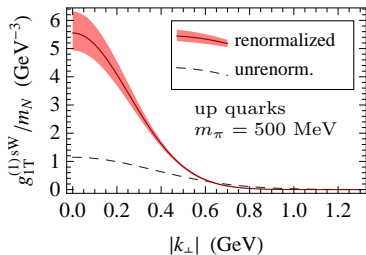
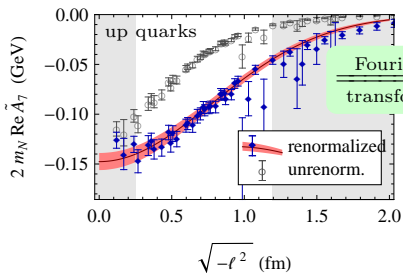
Backup Slides

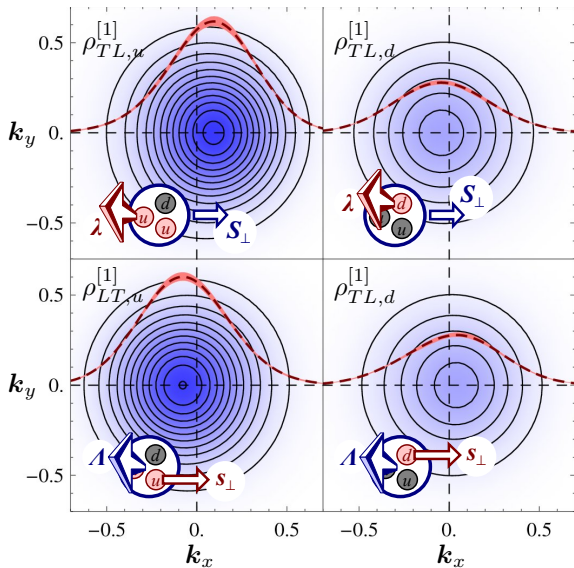




multiplicative renormalization constant Z adjusted to number of valence quarks
 $\int d^2 \mathbf{k}_\perp f_1^{(0)}(\mathbf{k}_\perp^2) = 2\tilde{A}_2(0, 0)$,
fixed in $u - d$ channel







Dipole deformations

$$\rho_{TL}^{[1]} : \sim \lambda \mathbf{k}_\perp \cdot \mathbf{S}_\perp g_{1T}$$

$$\rho_{TL}^{[1]} : \sim \Lambda \mathbf{k}_\perp \cdot \mathbf{s}_\perp h_{1L}^\perp$$

The corresponding dipole structures $\sim \lambda \mathbf{b}_\perp \cdot \mathbf{S}_\perp$, $\sim \Lambda \mathbf{b}_\perp \cdot \mathbf{s}_\perp$ for impact parameter densities (from GPDs) are ruled out by symmetries.

$$f^{(m_x, n_{\perp})} \equiv \int_{-1}^1 dx x^m \int d^2 \mathbf{k}_{\perp} \left(\frac{\mathbf{k}_{\perp}^2}{2m_N^2} \right)^n f(x, \mathbf{k}_{\perp}^2)$$

Let us assume the amplitudes \tilde{A}_i are sufficiently regular at $\ell^2 = 0$.

$$\begin{aligned} \langle \mathbf{k}_{\perp} \rangle_{\rho_{TL}^{[1]}} &= \lambda \mathbf{S}_{\perp} m_N \frac{g_{1T}^{1}}{f_1^{[1](0)}} = \\ \lambda \mathbf{S}_{\perp} m_N \frac{\tilde{A}_7(0, 0)}{\tilde{A}_2(0, 0)} &\stackrel{?}{=} \lim_{\ell^2 \rightarrow 0} \lambda \mathbf{S}_{\perp} m_N \frac{\tilde{A}_7(\ell^2, 0)}{\tilde{A}_2(\ell^2, 0)} \end{aligned}$$

All self-energies from the gauge link cancel on the RHS
(\Rightarrow no dependence on the renormalization condition).

Similar to weighted asymmetries from experiment (\rightarrow EIC):

$$A_{LT}^{\frac{Q_T}{m_N} \cos(\phi_h - \phi_S)} = 2 \frac{\langle \frac{Q_T}{m_N} \cos(\phi_h - \phi_S) \rangle_{UT}}{\langle 1 \rangle_{UU}} \propto \frac{\sum_q e_q^2 x g_{1T,q}^{(1\perp)}(x) D_{1,q}(z)}{\sum_q e_q^2 x f_{1,q}(x) D_{1,q}(z)}$$

$$f_1^{[1]}(\mathbf{k}_\perp^2) = C_0 \exp(-\mathbf{k}_\perp^2/\mu_0^2)$$

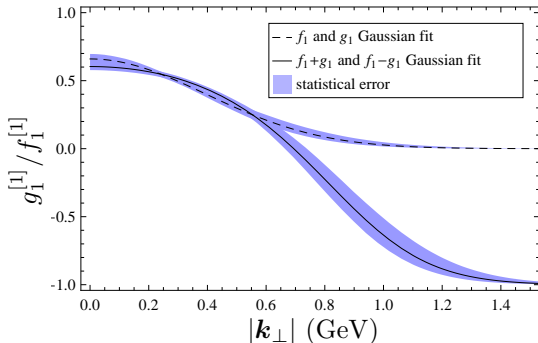
$$g_1^{[1]}(\mathbf{k}_\perp^2) = C_2 \exp(-\mathbf{k}_\perp^2/\mu_2^2)$$

vs.

$$\rho_{LL}^{[1]\pm}(\mathbf{k}_\perp) \equiv \frac{1}{2}f_1^{[1]}(\mathbf{k}_\perp^2) \pm \frac{1}{2}g_1^{[1]}(\mathbf{k}_\perp^2)$$

$$\rho_{LL}^{[1]+}(\mathbf{k}_\perp) = C_+ \exp(-\mathbf{k}_\perp^2/\mu_+^2)$$

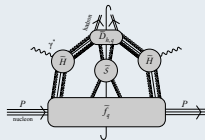
$$\rho_{LL}^{[1]-}(\mathbf{k}_\perp) = C_- \exp(-\mathbf{k}_\perp^2/\mu_-^2)$$



\Rightarrow Asymptotic behavior at large \mathbf{k}_\perp imposed by Gaussian ansatz; not a “lattice result”. Similar issues in analysis of experimental data.

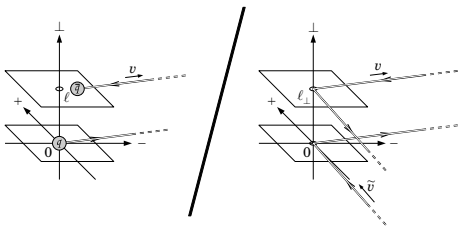
e.g., [JI, MA, YUAN PRD (2005)] :

$$W_{\text{unpol.,LO}}^{\mu\nu} \propto H \times f_1 \otimes D_h \otimes \underbrace{S}_{\text{soft factor}}$$



modified definition of TMD correlator:

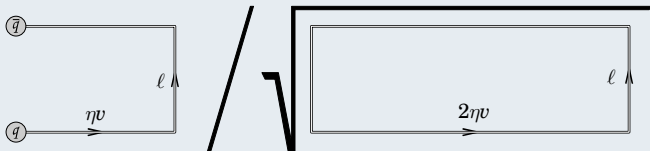
$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} e^{-ik \cdot \ell} \frac{\langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle}{\tilde{S}(\ell_{\perp}, \dots)}$$



- gauge links slightly off lightcone: $v \neq \hat{n}_{\perp}$
- \Rightarrow evolution eqn. in $\zeta \equiv (v \cdot P)^2 / v^2$
- soft factor \tilde{S} : vacuum expectation value of gauge link structure

How to get rid of the gauge link self energy $\exp(\delta m L)$?

Soft factor in TMD correlator? Suggestion [COLLINS arXiv:0808.2665] :



Is this a meaningful definition of TMDs?

prerequisite for quantitative lattice predictions

“To allow non-perturbative methods in QCD to be used to estimate parton densities, operator definitions of parton densities are needed that can be taken literally.” [COLLINS arXiv:0808.2665 (2008)]

k_{\perp} -moments from ratios of amplitudes ...

... bridge the gap until we know more.

Example Sivers effect: $\langle \mathbf{k}_{\perp} \rangle_{\rho_{TU}^{[1]}}$ from $\tilde{A}_{12}/\tilde{A}_2$.

Self-energies cancel, no explicit subtraction factor needed.

ratio of correlators far away from nucleon source and sink

$$\frac{C_{3\text{pt}}(\tau; \Gamma, \ell, P)}{C_{2\text{pt}}(P)} \xrightarrow{t_{\text{src}} \ll \tau \ll t_{\text{sink}}} \text{const. ("plateau value"),}$$

\downarrow
 access to $\langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle$

| | |
|------------------------------------|---|
| Γ | $\frac{1}{2} C_{3\text{pt}}^{\text{ren}}(\tau; \Gamma, \ell, P) / C_{2\text{pt}}(P)$ (LHPC projectors) |
| $\mathbf{1}$ | $\frac{m_N}{E(P)} \tilde{A}_1$ |
| $-\gamma_4 \gamma_5$ | $im_N \tilde{A}_7 \ell_z$ |
| γ_4 | \tilde{A}_2 |
| $\frac{1}{2} [\gamma_2, \gamma_4]$ | $\frac{1}{E(P)} \tilde{A}_9 P_x + \frac{im_N^2}{E(P)} \tilde{A}_{10} \ell_x + \frac{m_N^2}{E(P)} \tilde{A}_{11} (\ell_z)^2 P_x$ |
| ... | ... |

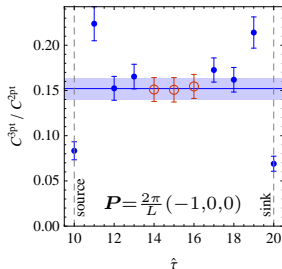
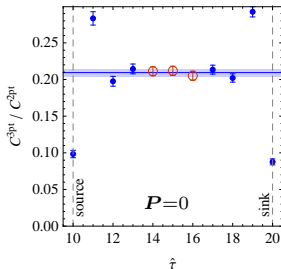
ratio of correlators far away from nucleon source and sink

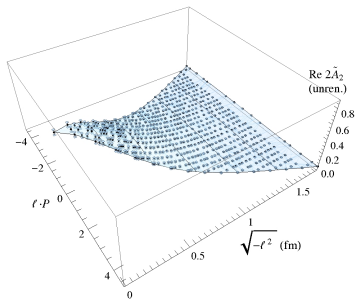
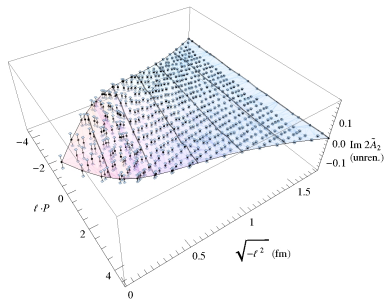
$$\frac{C_{3\text{pt}}(\tau; \Gamma, \ell, P)}{C_{2\text{pt}}(P)} \xrightarrow{t_{\text{src}} \ll \tau \ll t_{\text{sink}}} \text{const. ("plateau value"),}$$

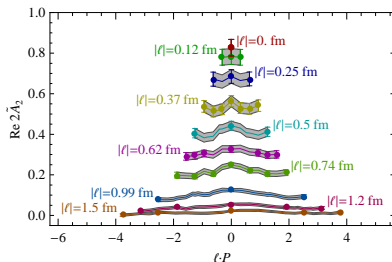
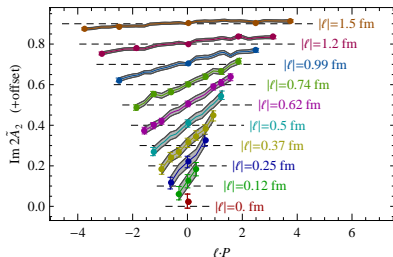
\downarrow
 access to $\langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle$

example plateau plots at $m_\pi \approx 600$ MeV

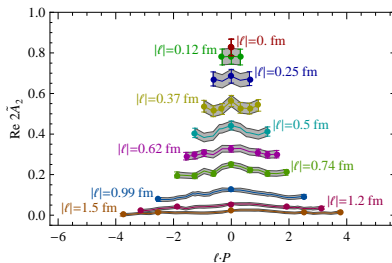
for $\Gamma = \gamma_4$ ($\Rightarrow \tilde{A}_2$), with HYP smeared gauge link $\mathcal{U} =$  :



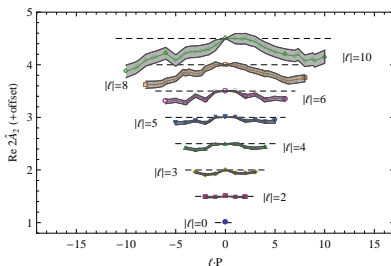
$2 \operatorname{Re} \tilde{A}_2(\ell^2, \ell \cdot P)$  $2 \operatorname{Im} \tilde{A}_2(\ell^2, \ell \cdot P)$ 

$2 \operatorname{Re} \tilde{A}_2(\ell^2, \ell \cdot P)$  $2 \operatorname{Im} \tilde{A}_2(\ell^2, \ell \cdot P)$ 

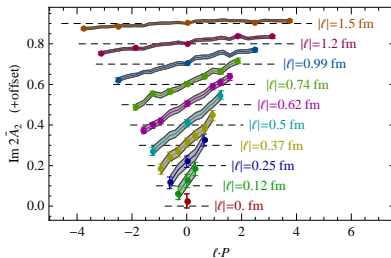
$$2 \operatorname{Re} \tilde{A}_2(\ell^2, \ell \cdot P)$$



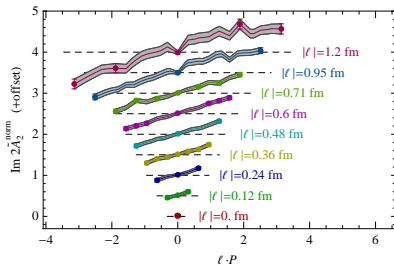
$$\operatorname{Re} \tilde{A}_2^{\text{norm}} = \frac{\operatorname{Re} \tilde{A}_2(\ell^2, \ell \cdot P)}{\operatorname{Re} \tilde{A}_2(\ell^2, 0)}$$



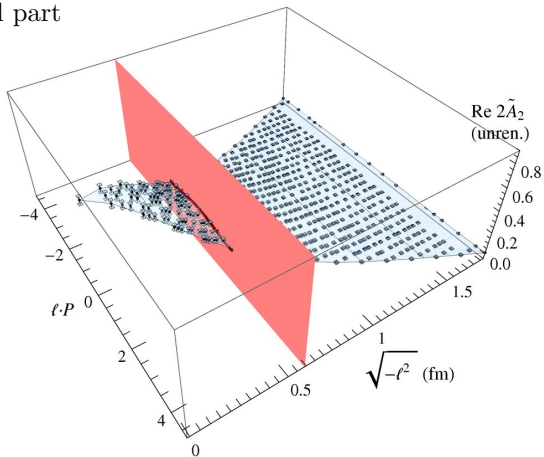
$$2 \operatorname{Im} \tilde{A}_2(\ell^2, \ell \cdot P)$$



$$\operatorname{Im} \tilde{A}_2^{\text{norm}} = \frac{\operatorname{Im} \tilde{A}_2(\ell^2, \ell \cdot P)}{\operatorname{Re} \tilde{A}_2(\ell^2, 0)}$$



real part



$$\ell^2 \xleftrightarrow{\text{FT}} k_{\perp}^2$$

$$\ell \cdot P \xleftrightarrow{\text{FT}} x$$

factorization hypothesis

$$f_1(x, \mathbf{k}_{\perp}^2) \approx f_1(x) f_1^{[1]}(\mathbf{k}_{\perp}^2) / \mathcal{N}$$

as in phenomenological applications,
e.g., Monte Carlo event generators

Then \tilde{A}_2 factorizes, too:

$$\tilde{A}_2(\ell^2, \ell \cdot P) = \tilde{A}_2^{\text{norm}}(\ell \cdot P) \tilde{A}_2(\ell^2, 0).$$

To test this, we define

$$\tilde{A}_2^{\text{norm}}(\ell^2, \ell \cdot P) \equiv \frac{\tilde{A}_2(\ell^2, \ell \cdot P)}{\text{Re } \tilde{A}_2(\ell^2, 0)}$$

(needs no renormalization!)

If factorization holds, $\tilde{A}_2^{\text{norm}}$ should be ℓ^2 -independent.

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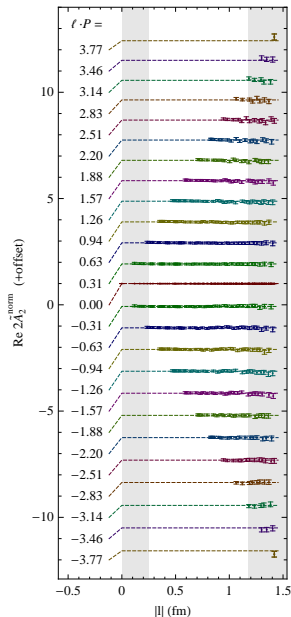
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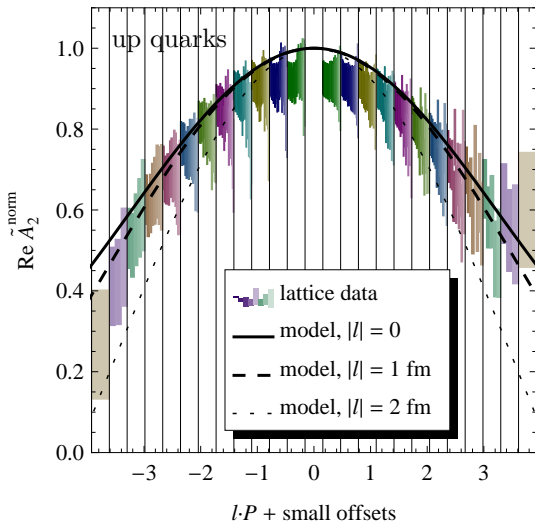
within statistics



All our data for $\tilde{A}_2^{\text{norm}}(\ell^2, \ell \cdot P)$ at $m_\pi \approx 600$ MeV

qualitative comparison to a scalar diquark model

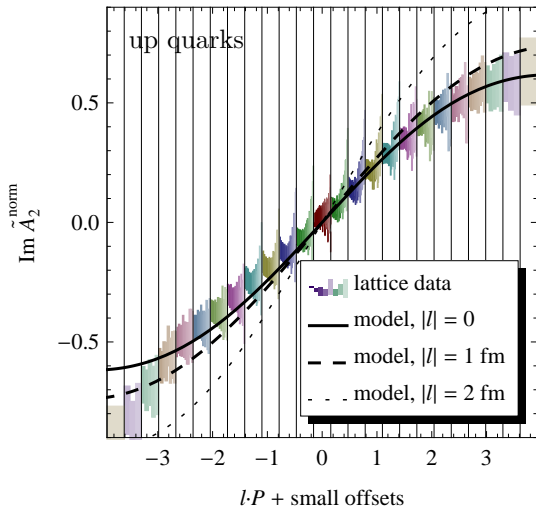
[BACCHETTA, CONTI, RADICI PRD (2008)] at $\sqrt{-\ell^2} = 0, 1$ and 2 fm

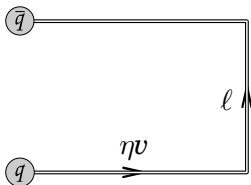


All our data for $\tilde{A}_2^{\text{norm}}(\ell^2, \ell \cdot P)$ at $m_\pi \approx 600$ MeV

qualitative comparison to a scalar diquark model

[BACCHETTA, CONTI, RADICI PRD (2008)] at $\sqrt{-\ell^2} = 0, 1$ and 2 fm





32 Lorentz-invariant amplitudes [GOEKE,METZ,SCHLEGEL PLB618,90 (2005)]

$$A_i\left(k^2, k \cdot P, \frac{v \cdot k}{|v \cdot P|}, \frac{v^2}{|v \cdot P|^2}, \frac{v \cdot P}{|v \cdot P|}\right) = A_i\left(k^2, k \cdot P, \underbrace{\frac{v \cdot k}{|v \cdot P|}}_{\approx x}, \zeta^{-1}, \text{sgn}(v \cdot P)\right)$$

Links approaching light cone: $v \rightarrow \hat{n}_- \Rightarrow \zeta \rightarrow \infty$. For large ζ , the evolution with ζ is known [COLLINS,SOPER NPB194,445 (1981)].

$$\left. \begin{array}{l} (v^0, v^1, v^2, v^3) \\ \text{future pointing } v \\ \text{TMDs for SIDIS} \end{array} \right\} \xrightarrow{\mathcal{T}} \left\{ \begin{array}{l} (-v^0, v^1, v^2, v^3) \\ \text{past pointing } v \\ \text{TMDs for Drell-Yan} \end{array} \right.$$

The transformation property of the matrix elements under time reversal provides relations:

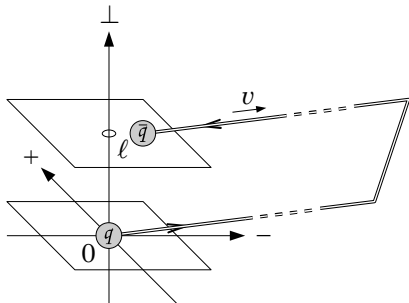
Example of a \mathcal{T} -even amplitude:

$$\begin{aligned} A_2\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, 1\right) &= A_2\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, -1\right) \\ &\Downarrow \\ f_1^{(\text{SIDIS})}(x, \mathbf{k}_\perp; \zeta, \dots) &= f_1^{(\text{Drell-Yan})}(x, \mathbf{k}_\perp; \zeta, \dots) \end{aligned}$$

Example of a \mathcal{T} -odd amplitude: (\rightarrow Siverson function f_{1T}^\perp)

$$\begin{aligned} A_{12}\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, 1\right) &= -A_{12}\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, -1\right) \\ &\Downarrow \\ f_{1T}^{\perp(\text{SIDIS})}(x, \mathbf{k}_\perp; \zeta, \dots) &= -f_{1T}^{\perp(\text{Drell-Yan})}(x, \mathbf{k}_\perp; \zeta, \dots) \end{aligned}$$

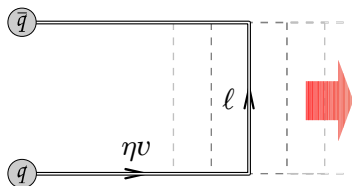
- ... appear in factorized SIDIS / Drell-Yan process
- are responsible for “time-reversal-odd” TMDs, such as f_{1T}^\perp (Sivers-function)



- gauge link = effective representation of struck quark (“final state interaction”)
- \Rightarrow (almost lightlike)

$$\zeta \equiv \frac{(v \cdot P)^2}{v^2} \rightarrow \pm \infty$$

- keep ζ finite to avoid “rapidity divergences”
- evolution equation in ζ [COLLINS, SOPER NPB (1981)]

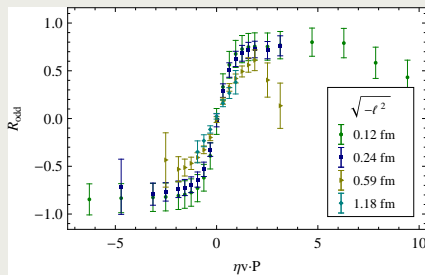


- v spatial $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |P_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes

[GOEKE, METZ, SCHLEGEL PLB (2005)]

$$\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta), \tilde{b}_i(\dots)$$

Test calculation: a time reversal odd ratio of amplitudes



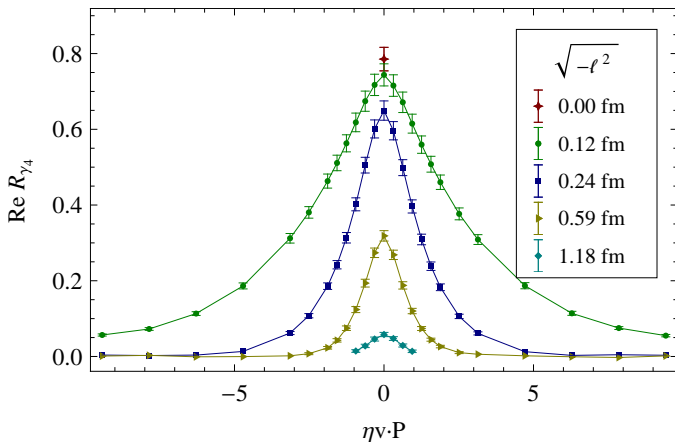
$$R_{\text{odd}} = -\frac{\tilde{a}_{12} - \left(\eta \frac{m_N^2 v_1}{P_1}\right) \tilde{b}_8}{\tilde{a}_2}$$

Plateaus visible at large $|\eta|$.

“Time-reversal odd” \leftrightarrow
odd in $\eta v \cdot P$.

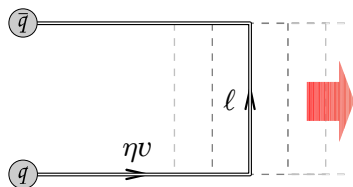
Part of the effect comes from the Sivers function f_{1T}^\perp !

$$\tilde{A}_2 \left(\ell^2, \ell \cdot P, \frac{v \cdot \ell}{|v \cdot P|}, \zeta^{-1}, \text{sgn}(v \cdot P) \right) \equiv \lim_{\eta \rightarrow \infty} \tilde{a}_2(\ell^2, \ell \cdot P, \eta v \cdot \ell, -\eta^2, \eta v \cdot P)$$



But $\tilde{a}_2 = \text{Re } R_{\gamma_4}$ always vanishes for large η !

Reason: power divergence suppresses $\tilde{a}_2 \sim \exp(-\delta m \eta)$.



- v spatial $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |P_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes $\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta)$

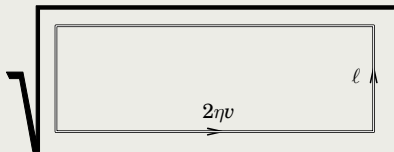
Problem: need to subtract gauge link self-energy ($\rightarrow \eta$ -independence)

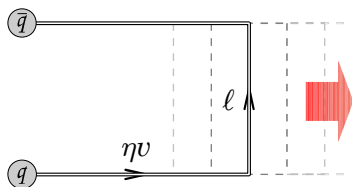
idea #1: modify definition of TMDs

[COLLINS PoS LC (2008)]

$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4 \ell}{(2\pi)^4} e^{-ik \cdot \ell} \frac{\langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle}{\tilde{S}(\ell_{\perp}, \dots)}$$

with \tilde{S} obtained from a vacuum expectation value of gauge links, e.g.,





- v spatial $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |P_{\text{lat.}}|^2$
- look for plateaus at large $|\eta|$
- now 32 amplitudes $\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta)$

Problem: need to subtract gauge link self-energy ($\rightarrow \eta$ -independence)

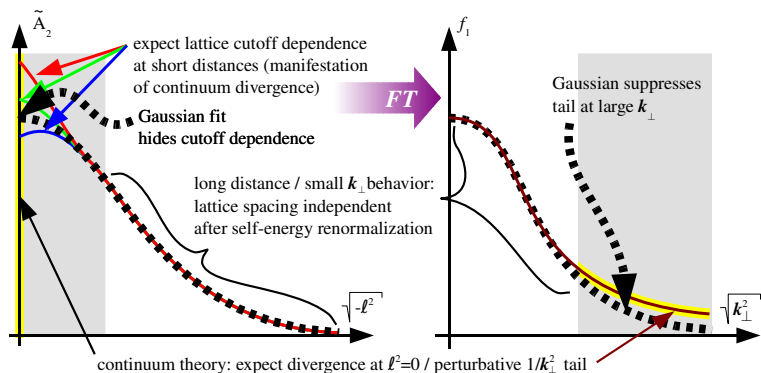
idea #2: ratios of amplitudes \rightarrow certain k_{\perp} -moments

e.g., formally,

$$\langle \mathbf{k}_y \rangle_{TU} = -2m_N \mathbf{S}_x \lim_{\eta \rightarrow \infty} \frac{\tilde{a}_{12}(0, 0, 0; \eta, \zeta) + \dots}{\tilde{a}_2(0, 0, 0; \eta, \zeta)} \propto \frac{\int dx \int d^2 \mathbf{k}_{\perp} \mathbf{k}_{\perp}^2 f_{1T}^{\perp}}{\int dx \int d^2 \mathbf{k}_{\perp} f_1}$$

Sivers function causes average transverse quark momentum in y -direction in a transversely polarized nucleon (spin in x -direction).

$$\langle \mathbf{k}_y \rangle_{TU} \underset{\eta \text{ large}}{\approx} -2m_N \mathbf{S}_x \frac{\tilde{a}_{12}(\ell_{\min}^2, 0, 0; \eta, \zeta) + \dots}{\tilde{a}_2(\ell_{\min}^2, 0, 0; \eta, \zeta)} \quad \text{Self-energy cancels!}$$



Problem with the perturbative tail

$\int d^2 \mathbf{k}_{\perp} f_1(x, \mathbf{k}_{\perp}^2)$ is undefined,
 in conflict with probability interpretation.

Gaussian is a poor man's solution.

Ideal would be a prescription that maintains

$$\int d^2 \mathbf{k}_{\perp} f_1(x, \mathbf{k}_{\perp}^2; \mu) = f_1(x; \mu) \quad \text{at some scale } \mu.$$